(c) $\varepsilon = 3 \Rightarrow$ hyperbola. Since $d = 2$

$$a = \frac{2}{9-1} = \frac{1}{4}, \quad b = \frac{2}{\sqrt{9-1}} = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}} \approx 0.7$$

$$c = \varepsilon a = \frac{3}{4}.$$

$$\longrightarrow \quad \frac{\left(x - \frac{3}{4}\right)^2}{\left(\frac{1}{4}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

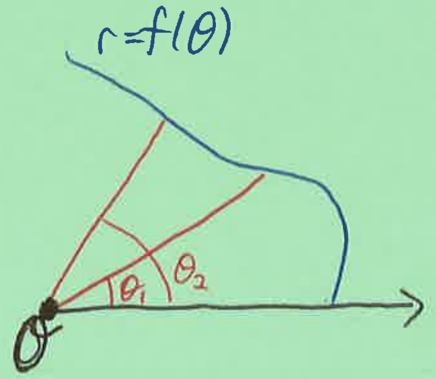
See next page for graph

Derivatives in Polar Coordinates

Let $r = f(\theta)$ be a function and $f'(\theta)$ its derivative.

Recall that, in Cartesian coordinates, the derivative gives the slope of the graph. What does the derivative mean in polar coordinates?

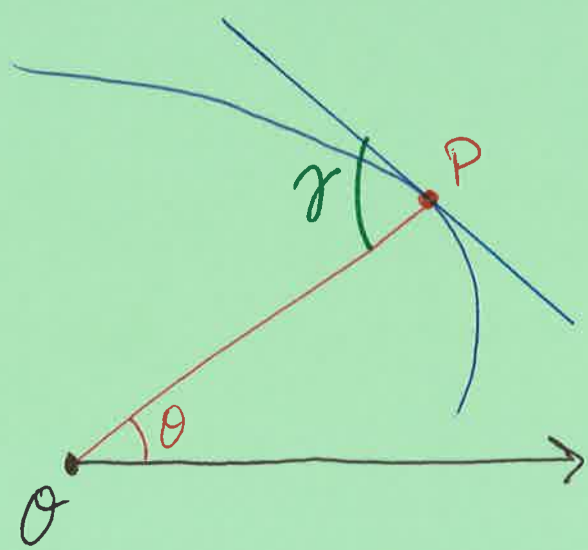
Consider a typical situation:



Suppose $f'(\theta) > 0$ over the interval $\theta_1 \le \theta \le \theta_2$. This means that $r = f(\theta)$ is increasing on this interval, i.e., moving further from $\mathcal{O}$.

Likewise, if $f'(\theta) < 0$, then $r = f(\theta)$ is decreasing as θ sweeps from θ_1 to θ_2.

Let's look at this another way now. Let $P = (f(\theta), \theta)$ be a point on the graph of $f(\theta)$ and assume $P \neq \mathcal{O}$, i.e., $f(\theta) \neq 0$. Let $\gamma = \gamma(\theta)$ be the angle, measured counterclockwise, from the tangent line to $f(\theta)$ at P to the line segment $\mathcal{O}P$.



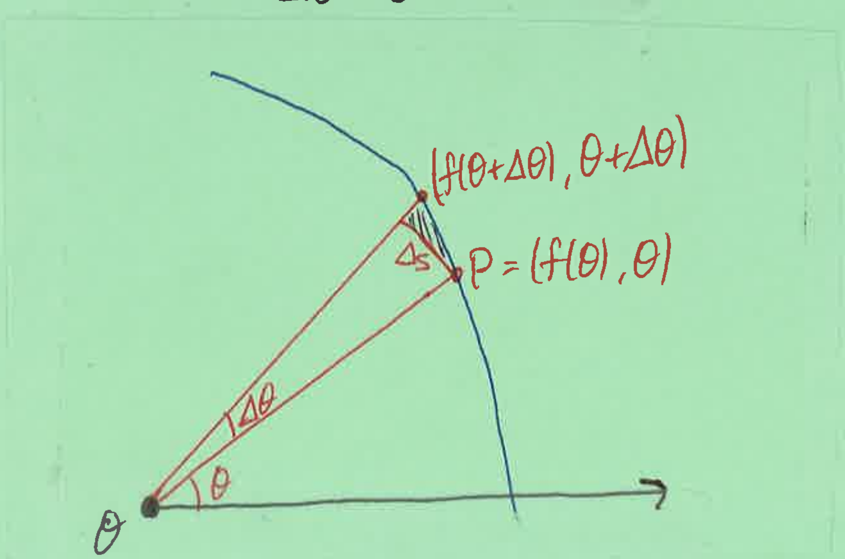
Notice that $0 \le \gamma < \pi$.

If $\gamma(\theta) > \frac{\pi}{2}$, then we can see that $f(\theta)$ is increasing at θ. If $\gamma(\theta) < \frac{\pi}{2}$, we see that $f(\theta)$ is decreasing at θ. If $\gamma(\theta) = \frac{\pi}{2}$, then $f(\theta)$ isn't changing at θ. Notice that these exactly correspond to the cases $f'(\theta) > 0$, $f'(\theta) < 0$, and $f'(\theta) = 0$, respectively. Thus, it is reasonable to expect a connection between $f'(\theta)$ & $\gamma(\theta)$...

So, what is this connection?

Definition of $f'(\theta)$:

$$f'(\theta) := \lim_{\Delta\theta \to 0} \frac{f(\theta + \Delta\theta) - f(\theta)}{\Delta\theta}$$



Let's call the (shaded) curved triangle the <u>beak</u> at P.

Recall that $\Delta s = f(\theta)\Delta\theta \Rightarrow \frac{1}{\Delta\theta} = f(\theta)\frac{1}{\Delta s}$.

So, the derivative is then:

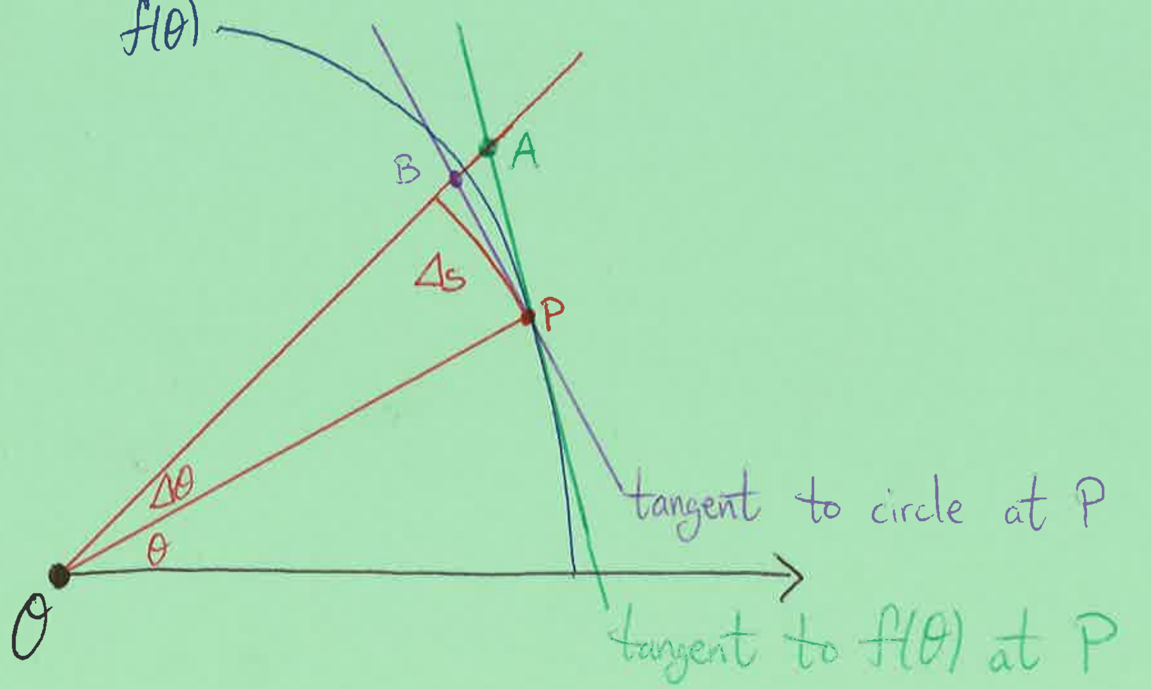
$$f'(\theta) = \lim_{\Delta\theta \to 0} \frac{f(\theta+\Delta\theta)-f(\theta)}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0} \frac{f(\theta+\Delta\theta)-f(\theta)}{\Delta s} \cdot f(\theta)$$

So, we need to understand

$$\lim_{\Delta\theta \to 0} \frac{f(\theta+\Delta\theta)-f(\theta)}{\Delta s}$$

Let's add some stuff to the previous graph



As we push $\Delta\theta$ to 0, the segment OBA rotates towards OP. Notice that as this happens, Δs approaches BP and the difference $f(\theta+\Delta\theta)-f(\theta)$ approaches AB. That is, the beak at P is approximated by ΔABP

Thus $\dfrac{f(\theta+\Delta\theta)-f(\theta)}{\Delta s}$ closes in on $\dfrac{AB}{BP}$.

Since $PB \perp P\mathcal{O}$, and $\mathcal{O}BA$ goes towards $P\mathcal{O}$, the angle $\angle PBA$ approaches $\frac{\pi}{2}$, hence ΔAPB becomes a right triangle, with right angle at B. Thus $\frac{AB}{BP}$ closes in on the tangent of the angle $\angle APB$. By definition, $\angle AP\mathcal{O} = \gamma$ & $\angle BP\mathcal{O} = \frac{\pi}{2}$, so $\angle APB = \angle AP\mathcal{O} - \angle BP\mathcal{O} = \gamma - \frac{\pi}{2}$

Thus, as $\Delta\theta \longrightarrow 0$,

$$\frac{f(\theta+\Delta\theta)-f(\theta)}{\Delta s} \longrightarrow \frac{AB}{BP} \longrightarrow \tan\left(\gamma - \frac{\pi}{2}\right)$$

Hence, finally,

$$\boxed{f'(\theta) = \tan\left(\gamma - \frac{\pi}{2}\right) \cdot f(\theta)}$$